The Existence of Equilibrium in Infinite-Dimensional Spaces: Some Examples

John H. Boyd III

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Abstract. This paper presents some examples that clarify certain topological and duality issues concerning the existence of equilibrium in infinite-dimensional spaces. One is a finite-dimensional version of an example due to Araujo (1985). It shows that equilibrium and individually rational Pareto optima can fail to exist even in finite-dimensional spaces if certain continuity conditions and compactness are not met. Araujo's example is not peculiar to infinite-dimensional economies. Re-examination of an example of Zame (1987) shows that economically reasonable equilibria may well exist even though prices fail to lie in the dual space specified by Zame. This suggests that the theorems of Zame and others be read as giving conditions for the existence of equilibrium prices in a particular dual, rather than for existence of equilibrium in general.

Department of Economics, University of Rochester, Rochester, NY 14627
Email: `boyj@troi.cc.rochester.edu (Internet)`

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1. Introduction

Much of the literature on equilibria in infinite-dimensional commodity spaces has pointed to the importance of using an appropriate topology and price space. However, rules for making that choice have not been explicitly spelled out. This has led to some confusion regarding the interpretation of some examples by Araujo (1985) and Zame (1987).

One of the alleged differences between finite and infinite dimensional spaces, the special role of the Mackey topology in infinite dimensional models, is illusory. Araujo's example is supposed to highlight this difference. In fact, a similar example can be constructed even when $\mathbb{R}^2$ is the commodity space. Araujo's results really warn against economies where there are no topologies that simultaneously yield a compact feasible set and upper semicontinuous preferences. Although this difficulty is more common in infinite-dimensional spaces, my example illustrates that there is nothing here that is peculiar to infinite-dimensional spaces.

A second stumbling block is whether to require that prices be in a previously given dual (e.g., the dual under the above topology). Some of Zame's examples of non-existence of equilibria do have equilibria, but the prices are not in Zame's (dual) price spaces. Nonetheless, these examples do have economically reasonable, non-dual, equilibrium prices that assign values to everything in the consumption sets.

In this paper I illustrate some of the main principles involved in choosing an appropriate commodity subspace, topologies and dual price space in the context of three examples. Two of these are well-known, one by Araujo (1985) and one by Zame (1987). The other example is a finite-dimensional version of Araujo's example.

2. Equilibrium Concepts

Any equilibrium model starts with two classes of agents—consumers and producers. Consumers make their choices among the bundles that they can both conceive of and afford, subject to survival constraints. They are not directly concerned with what is technically feasible to produce.[1] Producers are in the opposite circumstance—concerned with choosing among what is possible using their resources and technology, not what is conceivable or affordable to consumers.

In finite-dimensional models this distinction is adequately handled via consumption and production sets within a single ambient space, $\mathbb{R}^n$. In infinite-dimensional spaces there may be many plausible ambient spaces, each with many plausible topologies. Some analysis is required to determine the appropriate space and topology. Indeed, several spaces and topologies, used in different ways, may eventually be necessary. For example, Jones (1987) uses two different topologies in his existence theorem. Boyd and McKenzie (1993) use multiple price spaces, *ba* and the space of sequences of prices in $\mathbb{R}^m$.

Preferences and technology impose some restrictions on the appropriate spaces. At the very least, they should contain all that is feasible. We must be able to associate a price with any conceivable bundle, and highly-valued infeasible bundles should be too expensive for the consumer. There is no formal requirement that *all* bundles in the ambient space have well-defined prices, only those in the consumption and production sets.[2]

Zame (1987) has noted two types of problems.[3] The first obstacle to overcome is the existence of optimal allocations. This is typically accomplished by embedding the problem in some appropriate space, and then choosing a topology where the feasible set is compact

[1] This point was not sufficiently appreciated in the early literature. Bewley (1972) restricts the consumer to affordable bundles in ℓ^∞ rather than all affordable bundles.

[2] Even this can be weakened slightly, as in the equilibrium concept used by Boyd and McKenzie (1993).

[3] Jones (1986) has examines many of the major differences between models with finitely and infinitely many commodities.

and preferences are continuous. This is often a trivial step in finite-dimensional economies. In ℓ^∞, this puts very severe restrictions on preferences and technology, as pointed out by Araujo (1985). This has been assumed to be a special feature of infinite-dimensional spaces. In fact, the same sort of situation can easily arise even in $\mathbb{R}^2$, and is further examined in Section Three.

The second sticking point is the existence of supporting prices—for both consumption and production. This problem can be made less severe if we don't insist that prices lie in the dual of the space containing the consumption sets. This is illustrated in Section Four, where one of Zame's (1987) examples of non-existence of equilibrium is reconsidered. Although Zame is correct in maintaining that there are equilibrium prices in the dual, the example does have an obvious equilibrium.

3. The Existence of Pareto Optima

The first problem is the existence of individually rational Pareto optima. We start with a space large enough to contain the economically relevant consumption bundles—those that are feasible. We want a topology on that subspace which gives a compact feasible set while retaining continuity of preferences. Typically, an appropriate weak topology can guarantee compactness by the Banach-Alaoglu Theorem. Convex preferences allow considerable freedom here. Any topology between the weak and corresponding Mackey topologies has the same closed, convex sets. Upper semicontinuity of preferences in any of these topologies implies upper semicontinuity in all of them.

The Mackey topology is often portrayed as the correct topology to use, but there is nothing magic about it. Rather, its use is based on the fact that Mackey upper semicontinuity and weak upper semicontinuity are equivalent for convex preferences. The feasible set can easily fail to be Mackey compact, even if it is weakly compact. For example, the unit ball in ℓ^2 is weakly compact, but not Mackey (= norm) compact. The reason the Mackey topology is used is that it is the strongest topology where continuity implies weak upper semicontinuity (for convex preferences). It's easier to be continuous in the Mackey topology than in any other such topology.

The focus on compactness of the feasible set is the key to understanding Araujo's (1985) example. In fact, they allow us to recast his example in a finite-dimensional setting. He finds that the Mackey (ℓ^∞, ℓ^1) topology is appropriate for existence of equilibrium. Why? What is so special about this topology and duality?

The principles above provide the answer. We want the feasible set to be compact. In his case, compactness of the unit ball suffices. The Banach-Alaoglu Theorem informs us that the unit ball is weak (ℓ^∞, ℓ^1) compact. Preferences must be weakly upper semicontinuous. The weakest continuity condition that accomplishes this is Mackey (ℓ^∞, ℓ^1) upper semicontinuity. This insures that individually rational Pareto optima exist. (Full continuity is actually required for the existence of equilibria.)

When this kind of continuity fails, not only may equilibria fail to exist, but even individually rational Pareto optimal allocations may not exist. After reading Araujo's paper, one might conclude that infinite-dimensional spaces are quite different from $\mathbb{R}^n$ in this regard. This is not the case. A similar phenomenon can occur even in $\mathbb{R}^2$. Impatience only enters through the interpretation of the continuity condition in Araujo's model.[4] In $\mathbb{R}^2$, continuity is divorced from such considerations. What is important, as in many economic models, is that preferences be upper semicontinuous in a topology where the feasible set is compact. The example that I present only barely fails this continuity test. One agent has continuous preference on $\mathbb{R}^2_+$. The other agent's preferences are continuous on $\mathbb{R}^2_{++}$, but only lower semicontinuous on $\mathbb{R}^2_+$.

Let E and F denote locally convex vector spaces. If E is ordered, E_+ is its positive cone. Denote the weak, Mackey and strong topologies induced on E by F as $\sigma(E, F)$, $\tau(E, F)$

[4] This interpretation is stressed in Brown and Lewis (1981).

and $\beta(E,F)$, respectively. When $E=\ell^\infty$, the t^{th} component of a vector x is denoted $x(t)$, and e_t denotes the vector with $e_t(s)=0$ for $s\neq t$ and $e_t(t)=1$. Araujo makes the following assumptions:

Araujo Assumptions. The following 5 conditions hold:

(1) For every i, $\precsim_i$ is a complete, transitive and reflexive preorder on ℓ^∞_+.

(2) For every i and every $x\in\ell^\infty_+$, $\{z\in\ell^\infty_+ : x\precsim_i z\}$ and $\{z\in\ell^\infty_+ : x\succsim_i z\}$ are τ-closed.

(3) For every i there exists a $t(i)$ with $x+\lambda e_{t(i)}\succsim_i x$ for all $x\in\ell^\infty$ and$\lambda>0$.

(4) There exists a constant $a>0$ with $\omega_i(t)\geq a$ for all i and t.

(5) For every i and every $x\in\ell^\infty_+$, $\{z\in\ell^\infty_+ : x\precsim_i z\}$ is convex.

Araujo's main results may be summarized in the following metatheorem.

Theorem 1. *Let τ be a topology with $\sigma(\ell^\infty,\ell^1)\subset\tau\subset\beta(\ell^\infty,\ell^1)$. Then the following are equivalent.*

(1) *Assumptions (1)–(5) imply an equilibrium exists.*

(2) *Assumptions (1), (2) and (5) imply an individually rational Pareto optimum exists.*

(3) $\tau\subset\tau(\ell^\infty,\ell^1)$.

Araujo constucts an example to show (1) implies (3) as follows. If τ is as in Theorem 1, and $\tau\neq\sigma(\ell^\infty,\ell^1)$, then there is a τ-continuous linear functional $p\geq 0$ that is purely finitely additive. Let $\omega_1=\omega_2$ obey assumption (4) with $\omega_{11}=1$. Set $u_1(x_1)=x_{11}+p\cdot x_1$ and $u_2(x_2)=r\cdot x_2+p\cdot x_2$ where $r\in\ell^1_+$, $r_t>0$ for $t>1$. Notice that agent 1 does not value any specific goods from period 2 onwards. He only cares about them asymptotically because p is purely finitely additive. Let $\{\bar{x}_1,\bar{x}_2\}$ be a Pareto optimal allocation. Consumption at any specific time $t>1$ can be costlessly transferred to agent 2, who benefits from it. Thus $\bar{x}_{1t}=0$ for $t>1$. As there are 2 units of consumption at time 1 available, $u_1(\bar{x}_1)\leq 2$. But $u_1(\omega_1)=\omega_{11}+p\cdot\omega_1>2$, so any Pareto optimum is not individually rational, and hence there cannot be an equilibrium.

In Araujo's example, most goods are valueless to one of the agents, but that agent must have a non-zero amount of one of them to have positive utility. The other agent values all goods. Transferring some, but not all, of these valueless goods always results in a Pareto improvement. As they cannot all be transferred without making the first agent worse off, the Pareto optimum cannot be attained while maintaining individual rationality.

A similar phenomenon can even occur in $\mathbb{R}^2$. Define utility functions by $u_1(x,y) = y$ if $x > 0$ and $u_1(x,y) = 0$ if $x = 0$ and $u_2(x,y) = x+y$. Note that u_1 is concave since if $x > 0$ or $x' > 0$, $u(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y') = \alpha y + (1-\alpha)y' \geq \alpha u(x,y) + (1-\alpha)u(x',y')$, while if $x = x' = 0$, $u(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y') = 0 = u(x,y) + (1-\alpha)u(x',y')$. Give $\mathbb{R}^2$ the discrete topology. Both utility functions are continuous, and both are weakly monotonic. It follows that assumptions (1)–(3) are satisfied. Let the endowments be $\omega_1 = \omega_2 = (1,1)$. (Any strictly positive endowments will do.) Then Araujo's (4) is satisfied as well. Like the Araujo example, this exchange economy has no individually rational Pareto optimal allocations. If the allocation $\{(x_1,y_1),(x_2,y_2)\}$ is individually rational, then $u_1(x_1,y_1) \geq 1$. Thus $y_1 > 0$. Now consider the allocation $\{(x_1,y_1/2),(x_2,y_2+y_1/2)\}$. This is a Pareto improvement over the original allocation. Of course, this economy cannot have an equilibrium.

When the usual topology is applied to $\mathbb{R}^2$, this example just barely violates Araujo's assumptions. Agent 1's utility function is continuous on the interior of the positive orthant, and lower semicontinuous on the entire positive orthant. Upper semicontinuity fails only at the lower edge of the positive orthant. Agent 2's preferences are continuous in the usual topology. This seemingly small failure of continuity on part of the boundary destroys any possibility of an equilibrium.

It's easy to find conditions that imply the existence of individually rational Pareto optima on more general spaces. Let E be an arbitrary locally convex space. Weaken Araujo's assumptions as follows.

Modified Araujo Assumptions. The following three conditions hold:

(1*) For every i, $\precsim_i$ is a transitive and reflexive preorder.

(2*) For every i and every $x \in E_+$, $\{z \in E_+ : x \precsim_i z\}$ is τ-closed.

(5*) For every i and every $x \in E_+$, $\{z \in E_+ : x \precsim_i z\}$ is convex.

Theorem 2. *Suppose the set of feasible allocations* $\mathbf{F}$ *is* τ*-compact, and that assumptions (1*), (2*) and (5*) are satisfied. Then an individually rational Pareto optimum exists.*

Proof. Given allocations x and y, define $x \precsim y$ if $x_i \precsim_i y_i$ for every household i. Let $\mathcal{S}$ be the collection of all y with $y \succsim \omega$. Define $\mathcal{P}(y) = \{x \in \mathbf{F} : y \precsim x\}$. Let $\mathcal{R}$ be a chain in $\mathcal{S}$. If $\mathcal{U}$ is a finite subset of $\mathcal{R}$, $\cap_{y \in \mathcal{U}} \mathcal{P}(y) = \mathcal{P}(\max \mathcal{U})$ is non-empty by transitivity. Thus $\{\mathcal{P}(y) : y \in \mathcal{R}\}$ has the finite intersection property. Since each $\mathcal{P}(y_i)$ is compact, $\cap_{y \in \mathcal{R}}$ is also non-empty. Any element of the intersection is an upper bound for $\mathcal{R}$. By Zorn's Lemma, there is a maximal element of $\mathcal{S}$. This is the desired individually rational Pareto optimum. □

Corollary 1. *Consider* ℓ^∞. *Suppose assumptions (1*), (2*) and (5*) are satisfied with* $\sigma(\ell^\infty, \ell^1) \subset \tau \subset \tau(\ell^\infty, \ell^1)$ *then an individually rational Pareto optimum exists.*

Proof. First note that $\mathbf{F} \subset B(0, \|\omega\|)$ is $\sigma(\ell^\infty, \ell^1)-$compact by the Banach-Alaoglu Theorem. Since preferences are convex, the sets $\mathcal{P}(y)$ are $\sigma(\ell^\infty, \ell^1)-$closed. Hence the theorem applies. □

Under some additional assumptions, Aliprantis, Brown and Burkinshaw (1987a, b) show that compactness of the feasible set implies the core is non-empty.[5]

4. Equilibrium with Non-Dual Prices

I will only consider Zame's first example, but similar considerations apply to his other examples. In his first example, the consumption set is ℓ^1_+. There is one consumer with

[5] Boyd and McKenzie (1993) show a similar result for capital accumulation models with survival constraints.

utility function $u(X) = \sum_{t=1}^{\infty} x(t)$ and endowment $\omega = (2^{-2t})$. The production set is the cone generated by the negative orthant and vectors of the form $-e_k + 2e_{k+1}$ where k is not a power of two. As a result, no goods may be carried over to period $2^s + 1$ from period 2^s.

Let $0 < k \leq 2^{s-1}$ and consider $t = k + 2^{s-1}$. Then $c(t) \leq \sum_{j=1}^{k} 2^{k-j}\omega(j + 2^{s-1}) = \sum_{j=1}^{k} 2^{(k-3j-2^s)} = \frac{1}{7}2^{(k-2^s)}[1 - 2^{-3k}]$. It follows that $c(t)$ grows no faster than $z(t) = 2^{(k-2^s)}$. The feasible set is contained in the Riesz ideal $\mathbf{Z}$ generated by z. The space $\mathbf{Z}$ is the Riesz ideal recommended by Aliprantis, Brown and Burkinshaw (1987a, b). As Zame notes, any equilibrium price system π must obey $\pi(t) \geq 2\pi(t+1)$ for t not a power of 2 and $\pi(2^s) = \pi(2^{s+1})$. This can be accomplished by setting $\pi(k+2^{s-1}) = 2^{(2^{s-1}-k)}$ for $0 < k \leq 2^{s-1}$ (note $\pi(2^s) = 1$). Thus $\pi(t)z(t) \leq 2^{-2^{s-1}}$ when $2^{s-1} < t \leq 2^s$. As this is summable, π is a linear functional on $\mathbf{Z}$, even though it is not continuous on all of ℓ^1.

Although not a linear functional, it still makes sense to think of this as a price system. Some bundles in ℓ^1_+ may have positively infinite price, but these are outside the budget set. The affordable bundles must be in ℓ^1 since $\pi(t) \geq 1$. Utility makes sense on the entire budget set. It is easy to see that the consumer only consumes at $t = 2^s$, but is otherwise indifferent about timing of consumption. The material balance condition ties it down so $c_t = 2^{-1-2^{s-1}}[8/7 - 2^{2^{s-1}}/7]$ in equilibrium.

Since Zame has imposed the conditions that prices must be bounded (for unclear reasons), he finds that there is no equilibrium. Nonetheless, the prices and allocations above yield a perfectly reasonable equilibrium. The producer maximizes profits over his feasible set, and the consumer maximizes utility over all conceivable, affordable consumption bundles.

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